Review of Electromagnetically Induced Transparency using the Density Operator and Quantum Field Theory picture

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Resumé(Rest of the project is in English)
En standard kvante-optisk behandling af "electromagnetically induced transparency" i et lambda system er blevet udført og vigtige teknikker indenfor AMO-fysik blev præsenteret. Dette inkluderer "the rotating frame", "the rotating wave approximation", "the density operator" og "the Master equation". Herefter blev en analytisk løsning af lambda-EIT systemet fundet i den statiske grænse. En numerisk beregning af systemets dynamik(tidsafhængighed) blev også fundet og denne stemte overens med den statiske løsning efter $t/\gamma \approx 20$.
I anden del af projektet bliver elektromagnetisme inden for kvantefeltteori præsenteret i form af QED. Herefter præsenteres et muligt Lagrange led for dipol-vekselvirkning mellem fermioniske atomer og det elektromagnetiske felt. Det vises at dette led reducerer til den normale dipol interaktion fra kvantemekanik i den ikke-relativistiske grænse. Rydberg polaritons blev behandlet som masse-egentilstandene af den kvadratiske Hamiltonian for "ladder EIT" systemet. Vekselvirkningen mellem Rydberg polaritons blev beskrevet ved et quartic led. Resulterende Feynman diagrammer blev præsenteret og Bethe-Salpeter ligningen for vekselvirkningen blev udledt. Det blev vist, at i den korrekte grænse opfører systemet sig som en enkelt partikel der propagerer i et effektivt potentiale.

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1 Introduction

The phenomenon of electromagnetically induced transparency(EIT) happens when a medium becomes transparent for a probe field(typically lasers) after a control field changes the behaviour of the medium through an interference effect. EIT has been successfully modelled by standard methods in quantum optics [5]. So has the interactions between highly excited Rydberg atoms and its possible uses in quantum information [11]. Recently, a combination of the Rydberg interactions and slow light in the EIT setup [8] has been studied as a method of photon-photon interaction [10] [6] [9] and has promising uses in quantum information. Quantum field theory(QFT) approaches to describe these phenomena has been developed [2] [3] [7]. This motivates collaboration between high energy physicists who are experienced in such theory and quantum optics physicists who understand the physical systems and experiments. This project is meant to bridge the gap between the two communities[1]. The first part introduces the very basics of quantum optics in a review of standard EIT, and the second part is a naive approach to make a field theory for the Rydberg polaritons. It is heavily inspired by [2] and [3], but derived from more fundamental principles.

[1]Especially between the two collaborating groups Nonlinear Quantum Optics group and CP3 Origins at SDU.

Part I
Standard Electromagnetically Induced Transparency

A typical EIT setup uses two near resonant lasers that facilitate transitions between three states in an atom. Figure 1 shows the energy levels and laser frequencies.

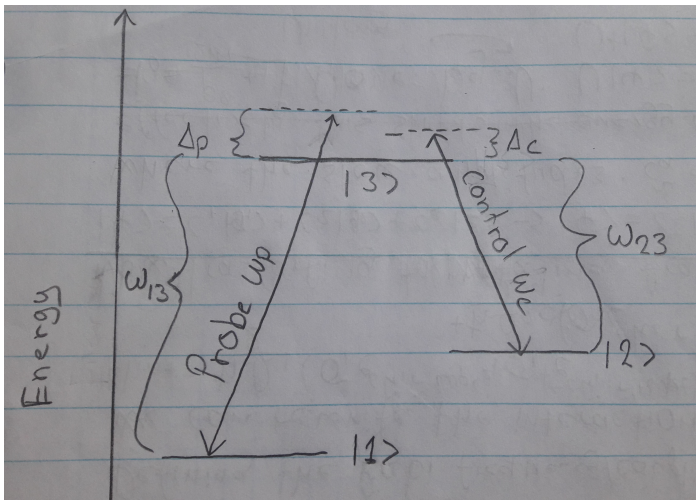


Figure 1: λ-type EIT setup

2 Bare Atom

When considering the bare atom without lasers, the system consists of an atom where only 3 states are relevant; two so called ground states: $|1\rangle$ & $|2\rangle$ and an exited state $|3\rangle$. We describe these energy levels by their resonant frequencies, such that the Hamiltonian for the atom becomes:

$$H_A = \hbar(\omega_{13} - \omega_{23}) |2\rangle \langle 2| + \hbar\omega_{13} |3\rangle \langle 3| \tag{2.1}$$

3 Laser Fields & Interaction

We turn on two laser fields(electromagnetic fields). We only care about the fields at a single point, so we do not describe their spacial dependence and their spacial frequency k_i. Letting $\hat{\epsilon_i}$ describe their direction and polarization, the fields are given by:

$$\vec{E}(\vec{r}, t) = \hat{\epsilon_1} E_1 \cos(-\omega_p t) + \hat{\epsilon_2} E_2 \cos(-\omega_c t) \tag{3.1}$$

These fields effectively adds an atom-field interaction term H_{AF} to the Hamiltonian of the atom, such that the total Hamiltonian becomes:

$$H = H_A + H_{AF} \tag{3.2}$$

This interaction is described through the dipole moment:

$$H_{AF} = -\hat{d}\ \vec{E} \tag{3.3}$$

Where the dipole moment operator is given by

$$\hat{d} = q\hat{r} \tag{3.4}$$

To calculate (3.3), we decompose the electric fields into "positively rotating" and "negatively rotating" parts $\hat{E} = \hat{E}^{(+)} + \hat{E}^{(-)}$. Where:

$$\vec{E}^{(\pm)} = \frac{1}{2}(\hat{\epsilon_1}E_1 e^{-i(\pm\omega_p t)} + \hat{\epsilon_2}E_2 e^{-i(\pm\omega_c t)}) \tag{3.5}$$

Similarly we want to decompose $\hat{d}$. Expanding the operator by applying $1 = (|1\rangle\langle 1| + |2\rangle\langle 2|)+|3\rangle\langle 3|))$ on both sides and assuming that states do not have any dipole by themselves and that $|2\rangle \to |1\rangle$ is not dipole-allowed, we get:

$$\hat{d} = |1\rangle\langle 1|\hat{d}|3\rangle\langle 3| + |2\rangle\langle 2|\hat{d}|3\rangle\langle 3| + H.c. \tag{3.6}$$

Choosing phase of the dipole operator such that $\langle i|\hat{d}|j\rangle$ is real, and defining operator $\sigma_{ij} = |i\rangle\langle j|$ we get:

$$\hat{d} = \langle 1|\hat{d}|3\rangle(\sigma_{13} + \sigma_{31}) + \langle 2|\hat{d}|3\rangle(\sigma_{23} + \sigma_{32}) \tag{3.7}$$

σ_{13} and σ_{23} are proportional to $e^{-i\omega_{13}}$ and $e^{-i\omega_{23}}$ while σ_{31} and σ_{32} are proportional to $e^{i\omega_{13}}$ and $e^{i\omega_{23}}$ (explained in section 6.1). Naming the positively rotating terms $\hat{d}^{(+)}$ and the negatively rotating $\hat{d}^{(-)}$ we can compute H_{AF} by letting the decomposed dipole operator act on the decomposed fields:

$$H_{AF} = -(\hat{d}^{(+)} + \hat{d}^{(-)})\ (\vec{E}^{(+)} + \vec{E}^{(-)}) \approx -\hat{d}^{(+)}\vec{E}^{(-)} - \hat{d}^{(-)}\vec{E}^{(+)} \tag{3.8}$$

The Rotating Wave Approximation(RWA) is performed, such that the terms that oscillate with double frequency is discarded($\hat{d}^{(+)}\vec{E}^{(+)} - \hat{d}^{(-)}\vec{E}^{(-)}$). This is reasonable because we are not interested in physics that occurs at these time scales. Instead only the cross terms survive.

Defining the Rabi couplings/ Rabi frequencies for the two fields:

$$\Omega_p = -\frac{E_1}{\hbar}\langle 1|\hat{\epsilon_1}\cdot\hat{d}|3\rangle \tag{3.9}$$

$$\Omega_c = -\frac{E_2}{\hbar} \langle 2| \hat{\epsilon_2} \cdot \hat{d} |3\rangle \tag{3.10}$$

And assuming that the energy levels spaced such that the fields only couple their respective transitions, we can write the interaction term in the RWA as:

$$H_{AF} = \frac{\hbar\Omega_p}{2}(\sigma_{13}e^{-i\omega_p t} + \sigma_{31}e^{i\omega_p t}) + \frac{\hbar\Omega_c}{2}(\sigma_{23}e^{-i\omega_c t} + \sigma_{23}e^{i\omega_c t}) \tag{3.11}$$

So now the total hamiltonian becomes:

$$H = H_A + H_{AF} = \hbar \begin{bmatrix} 0 & 0 & \frac{\Omega_p}{2}e^{-i\omega_p t} \\ 0 & \omega_{13} - \omega_{23} & \frac{\Omega_c}{2}e^{-i\omega_c t} \\ \frac{\Omega_p}{2}e^{i\omega_p t} & \frac{\Omega_c}{2}e^{i\omega_c t} & \omega_{13} \end{bmatrix} \tag{3.12}$$

4 Rotating Frame

To simplify calculations, we will transform to a different basis. Before transformation, the Hamiltonian was defined on the state vector ψ, which consists of the coefficients on the eigenvectors needed to produce the given state $|\psi\rangle$

$$\psi = \begin{bmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \end{bmatrix} \tag{4.1}$$

$$|\psi\rangle = c_1(t)\,|1\rangle + c_2(t)\,|2\rangle + c_3(t)\,|3\rangle \tag{4.2}$$

We now transform to the new basis, where two of the coefficients' phases rotate with time. Elements in the new basis is indicated by ˜.

$$\tilde{\psi} = \begin{bmatrix} c_1(t) \\ \tilde{c_2}(t) \\ \tilde{c_3}(t) \end{bmatrix} \tag{4.3}$$

$$|\tilde{\psi}\rangle = c_1(t)\,|1\rangle + \tilde{c_2}(t)e^{i(\omega_p - \omega_c)t}\,|2\rangle\,\tilde{c_3}(t)e^{i\omega_p t}\,|3\rangle \tag{4.4}$$

This new basis is called the "Rotating Frame" or "the interaction picture". This transformation is facilitated by the unitary transformation matrix U:

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i(\omega_p - \omega_c t)} & 0 \\ 0 & 0 & e^{i\omega_p t} \end{bmatrix} \tag{4.5}$$

When doing a time dependent change of basis, one needs to transform the Hamiltonian, such that it still satisfies the Schrödinger equation. If the transformation U is unitary, the new Hamiltonian $\tilde{H}$ is easily computed [12]:

$$\tilde{H} = UHU^\dagger + i\hbar(\partial_t U)U^\dagger \tag{4.6}$$

Computing the total Hamiltonian in the Rotating Frame yields:

$$\tilde{H} = \hbar \begin{bmatrix} 0 & 0 & \frac{\Omega_p}{2} \\ 0 & (\omega_{13} - \omega_p) - (\omega_{23} - \omega_p) & \frac{\Omega_c}{2} \\ \frac{\Omega_p}{2} & \frac{\Omega_c}{2} & \omega_1 - \omega_p \end{bmatrix} = \hbar \begin{bmatrix} 0 & 0 & \frac{\Omega_p}{2} \\ 0 & \Delta_p + \Delta_c & \frac{\Omega_c}{2} \\ \frac{\Omega_p}{2} & \frac{\Omega_c}{2} & \Delta_1 \end{bmatrix} \tag{4.7}$$

Remember that ω_p & ω_c represents the experimental values of the lasers, while ω_{13} & ω_{23} represents the actual differences in energylevels in the atom. We therefore call $\Delta_p = \omega_p - \omega_{13}$ & $\Delta_c = \omega_c - \omega_{23}$ the detunings of the system.

5 Eigenstates of Interaction Hamiltonian

The next step in many standard quantum mechanics problems would be finding the eigenstates of the system(ie diagonalizing the Hamiltonian). This is also possible for this system and leads to a very phenomenological understanding. This is not as useful and I will therefore just briefly introduce it. The simplest way to describe the eigenstates of the new Hamiltonian is using the "angles" ϕ & θ:

$$\tan\theta = \frac{\Omega_p}{\Omega_c} \tag{5.1}$$

$$\tan 2\phi = \frac{\sqrt{\Omega_p^2 + \Omega_c^2}}{\Delta_p} \tag{5.2}$$

When $\Delta_p = \Delta_c$, the eigenstates are given by

$$|0\rangle = \cos\theta\,|1\rangle - \sin\theta\,|2\rangle \tag{5.3}$$

$$|+\rangle = \sin\theta\sin\phi\,|1\rangle + \cos\phi\,|3\rangle + \cos\theta\sin\phi\,|2\rangle \tag{5.4}$$

$$|-\rangle = \sin\theta\cos\phi\,|1\rangle - \sin\phi\,|3\rangle + \cos\theta\cos\phi\,|2\rangle \tag{5.5}$$

The interesting point is now that the $|0\rangle$ state does not have any contribution from $|3\rangle$. This means that there is no possibility of spontaneous decay and thus scattering of the field and the medium becomes transparent. This offers a phenomenological explanation of EIT. But if one wants a more rigorous and useful description, the answer is to work with the density operator.

6 Density Operator Method

An often used method in AMO-physics(Atomic & Molecular Optics) is the density operator formalism. It uses the density operator $\hat{\rho}$. In a system where each point has the probabilities P_α of being prepared in the quantum states $|\psi_\alpha\rangle$, the density operator is mathematically described by

$$\hat{\rho} = \sum_\alpha P_\alpha |\psi_\alpha\rangle \langle\psi_\alpha| \tag{6.1}$$

If there is a state such that $P_\alpha = 1$, then the system is said to be pure. If not then the system is called mixed, and is a statistical mixture of quantum states. In our case P_α will always be unity. The density operator holds all the usual information about the state. Given the density operator for a state, one can compute the expectation value of an operator A using the trace:

$$\begin{aligned}
\langle A\rangle &= \langle\psi| A |\psi\rangle && (6.2)\\
&= \sum_\alpha P_\alpha \langle\alpha| A |\alpha\rangle && (6.3)\\
&= \sum_{\alpha,n} P_\alpha \langle\alpha| A |n\rangle \langle n|\alpha\rangle && (6.4)\\
&= \sum_{\alpha,n} \langle n|\alpha\rangle P_\alpha \langle\alpha| A |n\rangle && (6.5)\\
&= \sum_n \langle n| \sum_\alpha P_\alpha |\alpha\rangle \langle\alpha| A |n\rangle && (6.6)\\
&= \sum_n \langle n| \rho A |n\rangle && (6.7)\\
&= \mathrm{Tr}[A\rho] && (6.8)
\end{aligned}$$

As no assumptions were made of the orthonormal basis $|n\rangle$, the trace can be done with respect to any such basis.

6.1 Density Matrix and Coherences

Given an orthonormal basis, the density operator can be represented by a matrix(the density matrix). If $|n\rangle$ and $|n'\rangle$ are basis vectors of this basis, the entries of the density matrix are given by:

$$\rho_{nn'} = \langle n| \hat{\rho} |n'\rangle \tag{6.9}$$

The diagonal elements are easily interpreted. These are the probability that the system described by ρ are in the state $|n\rangle$:

$$\rho_{nn} = \langle n| \sum_\alpha P_\alpha |\alpha\rangle \langle\alpha|n\rangle = \sum_\alpha P_\alpha |\langle n|\alpha\rangle|^2 = P_n \tag{6.10}$$

Thus the diagonal elements satisfies unity of probability:

$$\sum_n \rho_{nn} = 1 \tag{6.11}$$

The off diagonal elements are harder to interpret. These describe the so called "coherences" of the system. Naively, one would think that we have simply chosen a bad basis, since the matrix is hermitian and thus can be diagonalized. This is true, but given the right basis, the diagonal entries can have insightful meaning. If one chooses the basis to be the states of the bare atom, the diagonal entries are related to transitions between states. Particularly an off diagonal element (e.g.ρ_{13}) is closely related to the dipole moment for that transition:

$$\hat{d_{probe}} = \langle 1|\, \hat{d}\, |3\rangle\, (\sigma_{13} + \sigma_{31}) \tag{6.12}$$

$$\left\langle \hat{d_{probe}} \right\rangle = \mathrm{Tr}\left(\hat{d_{probe}}\rho\right) = \mu_{13}(\rho_{13} + \rho_{31}) \tag{6.13}$$

Where $\mu_{ij} = \langle 1|\, \hat{d}\, |3\rangle$ is the transition dipole-moment, which usually is measured experimentally(it is possible to calculate for simple atoms and roughly estimate for more complicated ones).
This is a very relevant quantity when modelling the propagation of light.

7 Propagation of Light

Given the off-diagonal elements of a density matrix for a medium consisting of N atoms with identical density matrix in the volume V, one can compute the dipole moment for each atom. From this, one can compute the polarization of the medium:

$$P = \sum_i \frac{\left\langle \hat{d} \right\rangle_i}{V} = \frac{N}{V}(\mu_{13}\rho_{13} + \mu_{23}\rho_{23} + c.c.) \tag{7.1}$$

We are only interested in the polarization related to the probe. Using (7.1) and only including the term with ρ_{13} we get the positively rotating polarization related to the probe transition:

$$P^{(+)}_{probe} = N\mu_{31}\rho_{31} \tag{7.2}$$

Assuming that we are in the linear regime, the susceptibility is defined as:

$$P^{(+)} = N\mu_{13}\rho_{31} = \epsilon_0 \chi_{probe} E^{(+)} \tag{7.3}$$

Using our definition of the Rabi-frequency we get $E_{probe} = \omega_{13}\hbar/\mu_{31}$ and the susceptibility becomes:

$$\chi_{probe} = \frac{N\mu_{13}^2}{\epsilon_0 \hbar}\frac{2\rho_{31}}{\Omega_p} \tag{7.4}$$

This susceptibility holds almost all the information about the propagation of light in the medium. To see this we start with Mawell's equations:

7.1 Giving Meaning to the Susceptibility χ

Classically two of Maxwells equations describe light propagation:

$$\boldsymbol{\nabla} \times E = -\partial_t B \tag{7.5}$$

$$\boldsymbol{\nabla} \times B = \frac{1}{c^2}\partial_t(E + \frac{1}{\epsilon_0}P) \tag{7.6}$$

Where P is the polarization of the medium. In AMO physics we often do the Slowly Varying Envelope Approximation(SVEA), where we assume that the electric field and polarization are of the form[2]:

$$E(r,t) = \mathcal{E}(r,t)e^{i\vec{k}z - i\vec{\nu}t} + c.c \tag{7.7}$$

$$P(r,t) = \mathcal{P}(r,t)e^{i\vec{k}z - i\vec{\nu}t} + c.c \tag{7.8}$$

Where $\mathcal{E}\,\&\,\mathcal{P}$ vary slowly in $r\,\&\,t$. From this, the SVEA propagation equation can be derived:

$$\frac{1}{2ik}(\partial_x + \partial_y)\mathcal{E}(r,t) + \partial_z\mathcal{E}(r,t) = \frac{ik}{2\epsilon_0}\mathcal{P}(r,t) \tag{7.9}$$

Assuming the medium responds linearly($P = \epsilon_0\chi E$) we get:

$$\frac{1}{2ik}(\partial_x + \partial_y)\mathcal{E}(r,t) + \partial_z\mathcal{E}(r,t) = \frac{ik}{2}\chi\mathcal{E}(r,t) \tag{7.10}$$

Doing a Fourier transform from time to frequency-space we get the following equation for the envelope in frequency space($\mathcal{E}(\nu)$):

$$\frac{\partial\mathcal{E}}{\partial z} = \frac{i\nu}{c}\mathcal{E}(\nu) + \frac{i\vec{k}}{2}\chi(\nu)\mathcal{E} \tag{7.11}$$

Which is easy to solve:

$$\mathcal{E}(\nu,z) = \mathcal{E}(\nu,0)e^{i(\nu z/c + \vec{k}\chi(\nu))} \tag{7.12}$$

To get the solution in time-space, we transform back:

$$\mathcal{E}(t,z) = \int_{-\infty}^{\infty} d\nu\, e^{-i\nu t}\mathcal{E}(\nu,0)e^{i(\nu z/c + \vec{k}\chi(\nu))} \tag{7.13}$$

For monochromatic light with frequency ν_0 the envelope is $\mathcal{E}(\nu) \propto \delta(\nu - \nu_0)$. For this, we get the solution

$$\mathcal{E}(t,z) = \mathcal{E}e^{i\vec{k}\chi(\nu_0)z/2} \tag{7.14}$$

$$= \mathcal{E}e^{\vec{k}[Im(\chi(\nu_0)) + iRe(\chi(\nu_0))]z/2} \tag{7.15}$$

[2]If the light is not monochromatic the generalization (for E)is $\sum_n \mathcal{E}(r,t)e^{i\vec{k}z - i\vec{\nu}t} + c.c.$

From this we see that the imaginary part of $\chi(\nu_0)$ results in a change of the amplitude of the field. With the correct sign, we can thus relate the imaginary part of susceptibility to Transmission T

$$T = e^{Im(\chi)} \tag{7.16}$$

The real part effectively adds a phase shift to the field, which can be described by a change in the spacial frequency:

$$\vec{k} \to \vec{k} + \vec{k}\frac{Re(\chi)}{2} \tag{7.17}$$

8 Time Evolution and The Master Equation

For the usual state vector $|\psi\rangle$, the time evolution is described by the Schrödinger equation. An equivalent equation for the density operator can be derived from the Schrödinger equation:

$$\partial_t \rho = -\frac{i}{\hbar}[H, \rho] \tag{8.1}$$

This is the direct analogue to the Schrödinger equation. To envelop the effect of an environment on the states, one can use the so called "master equation". The master equation for a system interacting with a reservoir is given by [12]:

$$\partial_t \rho(t) = -\frac{i}{\hbar}\big[H_S + H_{SR}, \rho(t)\big] + \sum_\alpha k_\alpha D[S_\alpha, \rho(t)] \tag{8.2}$$

H_{SR} describes interaction between system and reservoir. The summation at the end primarily describes experimental "decoherences", which destroy the coherences represented by the off-diagonal elements in the density operator. In the our system, the master equation becomes:

$$\partial_t \rho = -\frac{i}{\hbar}[H_A + H_{AF}, \rho(t)] + \Gamma_{31} D[\sigma_{31}] + \Gamma_{32} D[\sigma_{32}] + \gamma_3 D[\sigma_{33}] + \gamma_2 D[\sigma_{22}] \tag{8.3}$$

The first term includes the total Hamiltonian already computed in (4.7). The 2nd and 3rd terms describes spontaneous decay from $|3\rangle \to |1\rangle$ and $|3\rangle \to |2\rangle$ respectively. The coefficient Γ_i is the strength of this decay. The last two terms describe the dephasing which removes coherences(off-diagonal elements in ρ) of the state.
All these terms are computed using the Lindblad super operator D which is defined as:

$$D[c, \rho] = c\rho c^\dagger - \frac{1}{2}[c^\dagger c\rho + \rho c^\dagger c] \tag{8.4}$$

We have now described an equation that governs the evolution of the physical system. This can be solved numerically for time evolution. Furthermore, it is possible to solve the stationary case symbolically.

9 Stationary Solution

After the EM-fields have been turned on for some time, the system will approximately be in a stationary equilibrium. Setting $\partial_t \rho(t) = 0$ in the left side of the master equation (8.3), we can solve this case symbolically. Using $\rho_{ij} = \rho^*_{ji}$ the problem simplifies to a cumbersome 6 equations with 6 variables. Mathematica can solve this as is, but with a few physically inspired approximations it can be solved neatly by hand.

9.1 Solution by Hand

In the case of a comparably weak probe field($|1\rangle - |3\rangle$ coupling), most of the population will be in the ground state. Since the few atoms that get exited will spontaneously decay back into the ground state fairly rapidly. Inspired by this, we set $\rho_{11} = 1$, $\rho_{22} = \rho_{33} = 0$. We also relabel the constants that go into the equations to fit the way they appear in the equations s.t. $\Gamma_3 = \Gamma_{31} + \Gamma_{32}$, $\gamma_{31} = \Gamma_3 + \gamma_3$, $\gamma_{32} = \Gamma_3 + \gamma_3 + \gamma_2$, $\gamma_2 = \gamma_2$. With this, the $\dot{\rho_{32}}$ equation in (8.3) yields:

$$\dot{\rho_{32}} = 0 = \Omega_p \rho_{12} + i(\gamma_{32} - 2i\Delta_c)\rho_{32} \tag{9.1}$$

$$\rho_{32} = \frac{i\Omega_p \rho_{12}}{\gamma_{32} - 2i\Delta_c} \tag{9.2}$$

Similarly $\dot{\rho_{21}}$ yields:

$$\dot{\rho_{21}} = 0 = -i\Omega_c \rho_{13} + i\Omega_p \rho_{32} - (\gamma_2 - 2i(\Delta_c - \Delta_p))\rho_{12} \tag{9.3}$$

$$\rho_{12} = \frac{-i\Omega_c \rho_{13}}{\gamma_2 - 2i(\Delta_c - \Delta_p)} \tag{9.4}$$

Where the term with ρ_{32} has been removed since inserting (9.2) results in a Ω_p^2 which is very small when the probe is weak.
$\dot{\rho_{31}}$ yields:

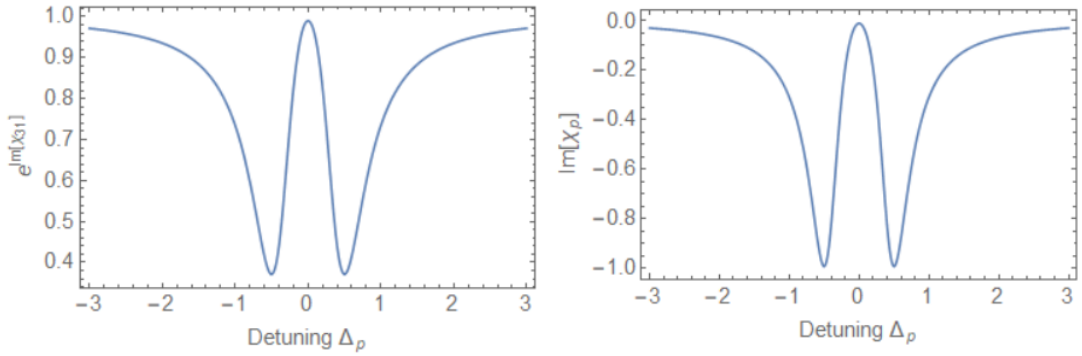
$$\dot{\rho_{31}} = 0 = \Omega_p + \Omega_c \rho_{21} + (\gamma_{31} - 2i\Delta_p)\rho_{31} \tag{9.5}$$

$$\rho_{31} = \frac{-i\Omega_c \rho_{21} + \Omega_p}{\gamma_{31} - 2i\Delta_p} \tag{9.6}$$

ρ_{31} is found by inserting $\rho_{21} = \rho^*_{12}$ from (9.4) and rearranging:

$$\rho_{31} = \frac{\Omega_p(\gamma_2 - 2i(\Delta_p - \Delta_c))}{(\gamma_{31} - 2i\Delta_p)(\gamma_2 - 2i(\Delta_p - \Delta_c))} \tag{9.7}$$

The susceptibility for the probe is then calculated using (7.4). With this, the transmission is computed from (7.16). Using units where $\gamma_{31} = 1,\ \Delta_c = 0,\ \Omega_c = 1,\ \Omega_p = 0.01$ & $\hbar = 1$, the transmission and $Im[\rho_{31}]$ is seen in figure 2



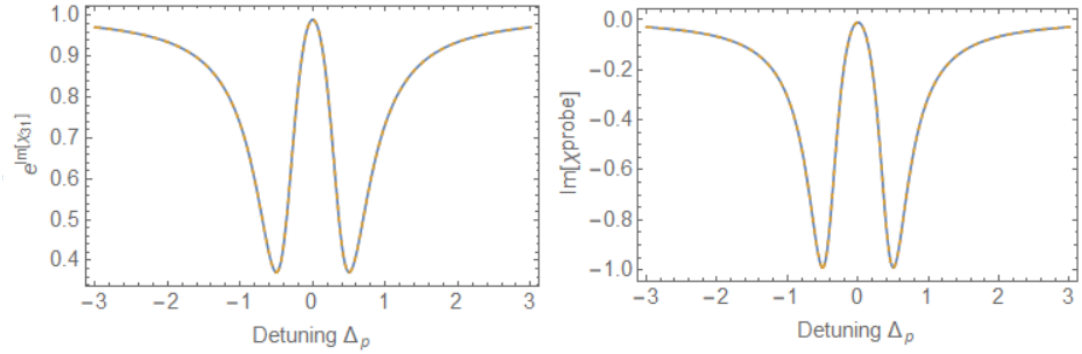
(a) Transmission for $\gamma_{31} = 1,\ \Delta_c = 0,\ \Omega_c = 1,\ \&\ \Omega_p = 0.01$

(b) Susceptibility for Transmission for $\gamma_{31} = 1,\ \Delta_c = 0,\ \Omega_c = 1,\ \&\ \Omega_p = 0.01$

Figure 2: By hand approximation

9.2 Full Stationary Solution

The full stationary equation((8.3) with $\partial_t \rho(t) = 0$) is possible to solve symbolically, but is very cumbersome.DSolve in Mathematica can solve this equation as is. The full solution is compared to the approximation for $\Omega_p/\Omega_c = 0.01$ in figure 3.

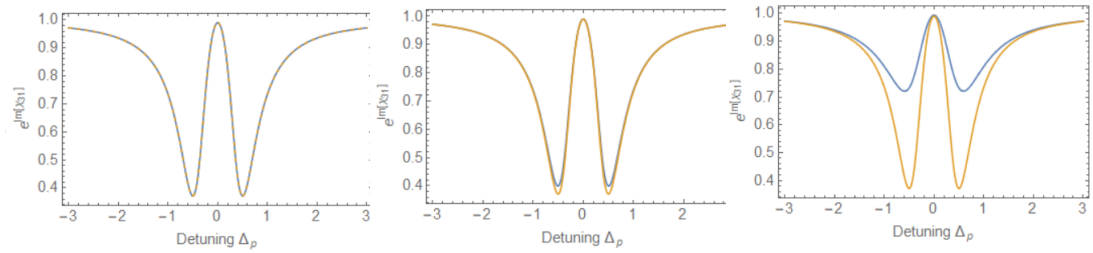


(a) Transmission for $\Omega_p = 0.01$. Full blue is the full solution and dashed orange is the approximation. The two solutions are indistinguishable.

(b) Susceptibility for $\Omega_p = 0.01$. Full blue is the full solution and dashed orange is the approximation. The two solutions are indistinguishable.

Figure 3: Comparison at $\Omega_p = 0.01$

For increasing Ω_p the approximation that $\Omega_p/\Omega_c \ll 1$ should fail. This is seen in figure 4 for $\Omega_p = 0.01$, $\Omega_p = 0.1$, $\Omega_p = 2$



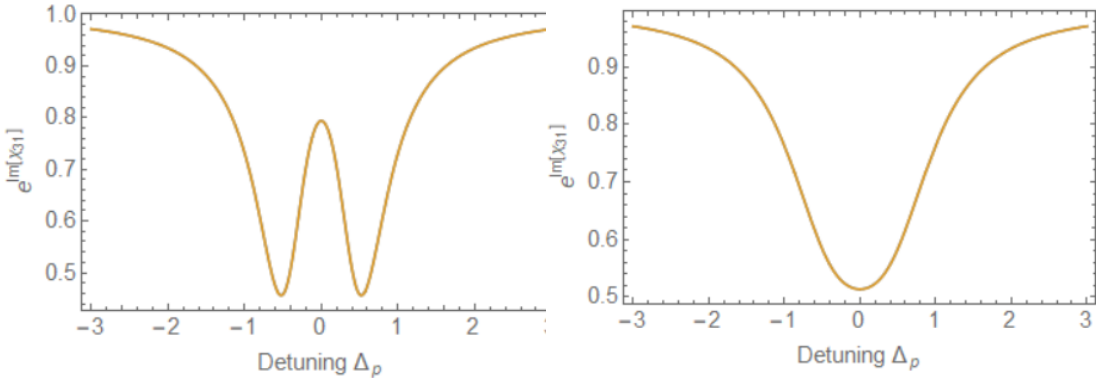
(a) Transmission for $\Omega_p = 0.01$ Blue is full solution and orange is approximation

(b) Transmission for $\Omega_p = 0.1$ Blue is full solution and orange is approximation

(c) Transmission for $\Omega_p = 0.5$ Blue is full solution and orange is approximation

Figure 4: Comparison for increasing Ω_p

Experimentally it is a challenge to preserve the coherences. For an experiment with a large decay of coherences out of $|2\rangle$, the EIT effect vanishes. This is shown in figure 5

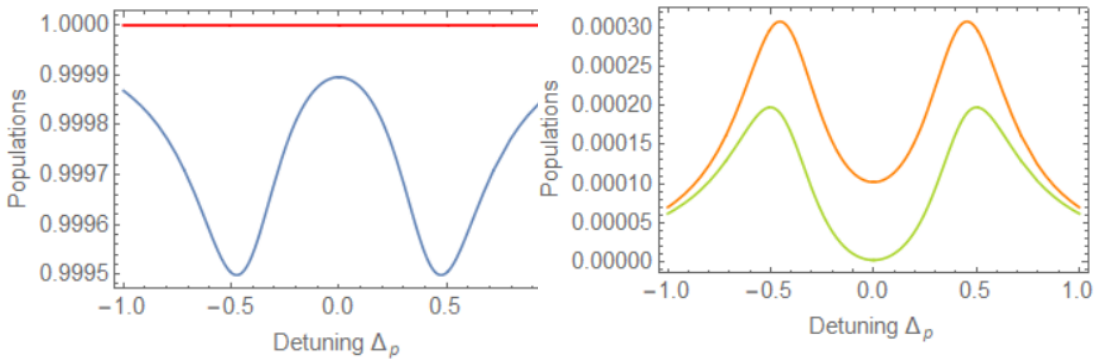


(a) $\gamma_2 = 0.3$

(b) $\gamma_2 = 2$

Figure 5: Transmission for large γ_2 decoherence

In the full solution the population of different states(ρ_{ii}) can also be computed. This is plotted as a function of detuning Δ_p in figure 6. We see that the approximation $\rho_{11} = 1$ & $\rho_{22} = \rho_{33} = 0$ hold for the used values of $\gamma_{31},\ \Delta_c,\ \Omega_c,\ \Omega_p$ & $\hbar = 1$.



(a) Population of in $|1\rangle$(blue) and total population(red)

(b) Population in $|2\rangle$ (orange) and $|3\rangle$(green)

Figure 6: Populations for different detuning Δ_p

Figure 7 shows how the populations approach these values, as we turn on the control field.

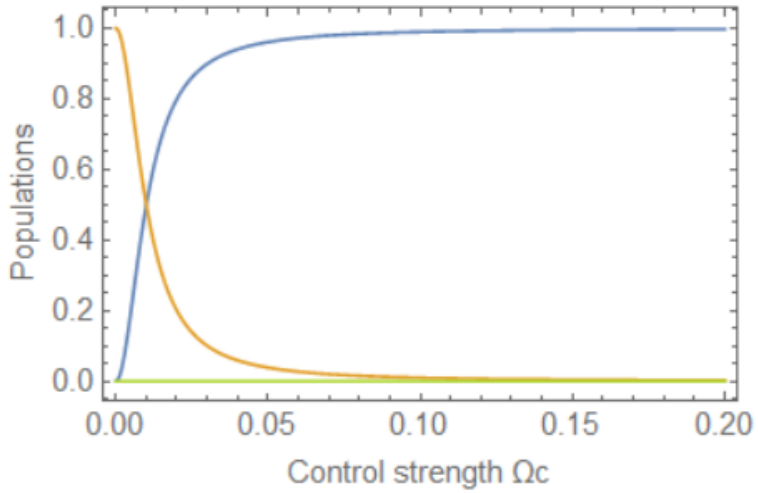
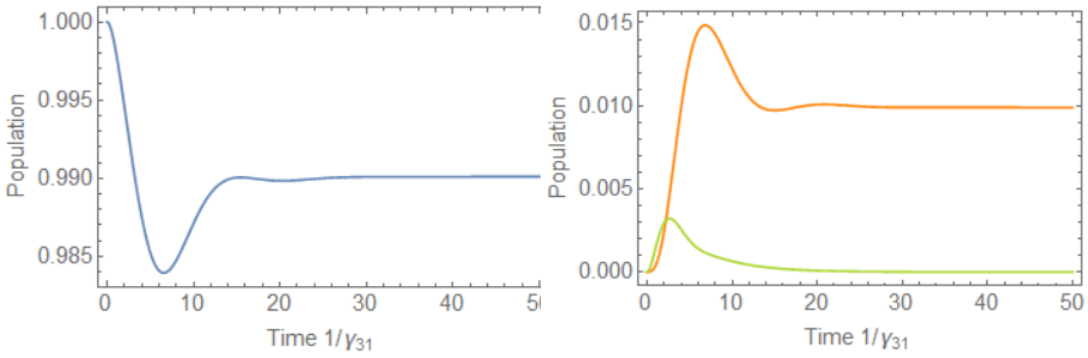


Figure 7: Population of $|1\rangle$(blue), $|2\rangle$(orange), $|3\rangle$(green)

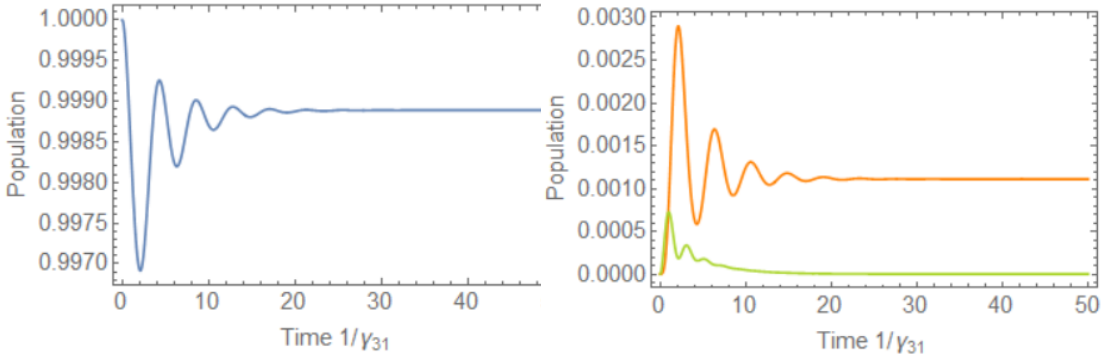
10 Full Numerical Solution

The full master equation can be solve numerically to find the time dependence of the system. The result is shown in figure 8 and 9



(a) $|1\rangle$ (b) $|2\rangle$(blue) and $|3\rangle$(orange)

Figure 8: Time dependence of populations for $\Omega_c = 1$



(a) $|1\rangle$ (b) $|2\rangle$(blue) and $|3\rangle$(orange)

Figure 9: Time dependence of populations for $\Omega_c = 3$

For higher Ω_c the system initially oscillates faster. These are the so called These are called Rabi-oscillations. In both cases, the overall dampening is the same, and the system reaches stability after some tine. So setting $\partial_t \rho = 0$ seems like a good approximation.
In figure 10 the time dependence of the imaginary part of ρ_{31} is plotted. This is proportional to the susceptibility. Here we also see stability after $t \approx 30 \cdot 1/\gamma_{31}$.

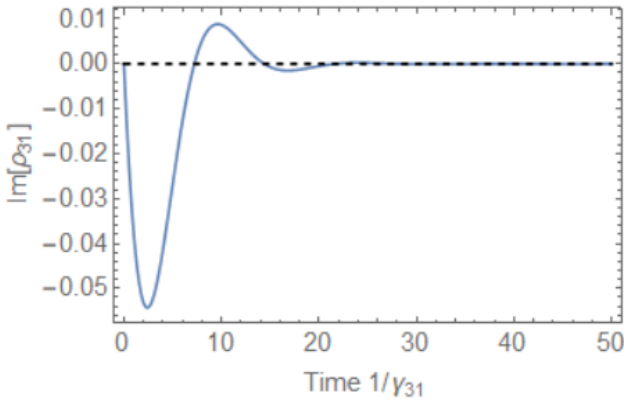


Figure 10: Time dependence of susceptibility. Blue is time dependent solution and dashed line is stationary approximation

Part II
Field Theory for Rydberg Polaritons

Now we move on to the field theory part of this project. We start by introducing the electric field and dipole interaction in quantum field theory(QFT). Then the EIT setup is treated with the field theory approach. This time the system will be the "ladder EIT setup"(figure 11). Finally we model Rydberg polaritons and their interactions.

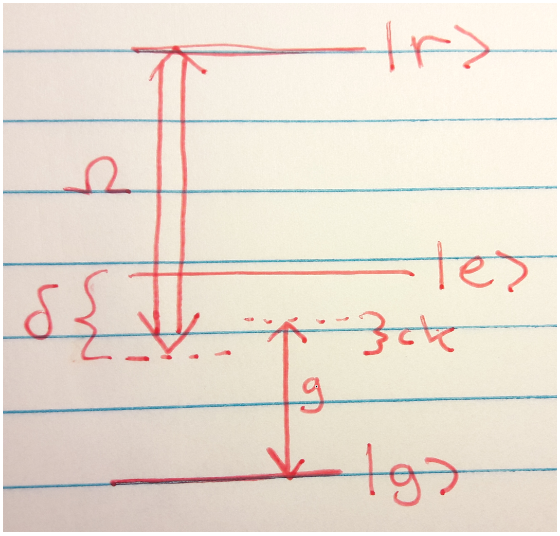


Figure 11: Ladder setup for EIT
The probe ω couples the Rydberg states $|r\rangle$and the exited state $|e\rangle$ with detuning δ. The probe with strength g couples the ground state and the exited state. It is "detuned" from the control field by the frequency ck.

11 Electric Field in QFT

In quantum field theory, the electric field is described together with the magnetic field in quantum electrodynamics, which has the free Lagrangian density(in absence of source currents):

$$\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{11.1}$$

Where Einstein summation notation is used [3]. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor. Where A_μ is vector field of electromagnetism. The components of $F_{\mu\nu}$ are the

[3]Whenever the same index is present twice in a term, it implies summation over this index. Greek letters symbolize spacetime quantities and run over $\{0,1,2,3\} = \{t,x,y,z\}$. Latin letters symbolize spacial quantities and run over $\{1,2,3\} = \{x,y,z\}$. In the above equation, summation is implied over μ and ν.

electric and magnetic fields:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{11.2}$$

To calculate the Hamiltonian related to this Lagrangian, we compute the conjugate momentum (π^μ)to A_μ:

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -F^{0\mu} = (0, E^i) \tag{11.3}$$

The Hamiltonian can then be computed:

$$H_{EM} = \int d^3x \pi^i \dot{A}_i - \mathcal{L} = \int d^3x \frac{1}{2} E_i E^i + \frac{1}{2}(\nabla_j A_i)(\nabla_i A_j) - A_0(\partial_i E^i) \tag{11.4}$$

There is some redundancy in our description of A_μ. It turns out one can choose whatever $\partial_i A^i$(divergence) of the field one wants. This is called Guage-freedom or Guage symmetry. A convenient choice for our case is Coulomb Guage where we set $\partial_i A^i = 0$ and as a result $A^0 = 0$. This removes the last term in (11.4):

$$H_{EM} = \int d^3x \frac{1}{2} E_i E_i + \frac{1}{2}(\nabla_j A_i)(\nabla_i A_j) \tag{11.5}$$

The electric field in QFT is the momentum conjugate to A_i. A_i is the field that satisfies the free part of the theory and is thus given by a (continuous) sum of momentum eigenstates.

$$\vec{A} = (\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{1}{2|\vec{p}|}} \sum_{r=1}^{2} \vec{\epsilon}_r(\vec{p}) \left[a^r_{\vec{p}} e^{ip_i x^i} + a^{r\dagger}_{\vec{p}} e^{-ip_i x^i} \right] \tag{11.6}$$

Where r denotes the polarisations that are present in ϵ_r. $a^r_{\vec{p}}$ are the annihilation operators that removes a particle with momentum $\vec{p}$. x^i labels the position in space. This is given in the Heisenberg picture where time dependence is included in the operators(like a's, $\vec{A}$ and $\vec{E}$). Using (11.3) calculating E from A is a simple time derivative, which will move a $|\vec{p}|$ down. This hints that the Coulomb Guage is not good at expressing Lorentz invariance. Though this is not a problem in our system, since we will treat it non-relativistically anyway.

$$\vec{E}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{|\vec{p}|}{2}} \sum_{r=1}^{2} \vec{\epsilon}_r(\vec{p}) \left[a^r_{\vec{p}} e^{ip_i x^i} - a^{r\dagger}_{\vec{p}} e^{-ip_i x^i} \right] \tag{11.7}$$

In experiments only a single polarisation is relevant. This polarisation is picked and the sum over r disappears. We start by calculating the electric part of the Hamiltonian

$$\frac{1}{2} \int d^3x \vec{E} \cdot \vec{E} = -\frac{1}{2} \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6} \frac{\sqrt{|\vec{p}||\vec{q}|}}{2} \vec{\epsilon}(\vec{p}) \cdot \vec{\epsilon}(\vec{q}) \tag{11.8}$$
$$\left(a_{\vec{p}} e^{ip_i x^i} - a^\dagger_{\vec{p}} e^{-ip_i x^i} \right) \left(a_{\vec{q}} e^{iq_i x^i} - a^\dagger_{\vec{q}} e^{-iq_i x^i} \right)$$

Assuming the polarizations are identical, $\vec{\epsilon}(\vec{p}) \cdot \vec{\epsilon}(\vec{q})$ vanishes. The integral over x^3 will produce delta functions from the exponentials.

$$= -\frac{1}{2}\int \frac{d^3p\, d^3q}{(2\pi)^6}\frac{\sqrt{|\vec{p}||\vec{q}|}}{2}\Big[a_{\vec{p}}a_{\vec{q}}\delta(\vec{p}+\vec{q}) \tag{11.9}$$
$$-a_{\vec{p}}a^\dagger_{\vec{q}}\delta(\vec{p}-\vec{q}) - a^\dagger_{\vec{q}}a_{\vec{p}}\delta(-\vec{p}+\vec{q}) + a^\dagger_{\vec{p}}a^\dagger_{\vec{q}}\delta(-\vec{p}-\vec{q})\Big]$$

Now we use $|\vec{p}| = \hbar\omega_p/c$ and introduce the terms that would originate from the $\nabla_i A_j \nabla_j A_i$ part of the Hamiltonian. These terms would be the same, except that spacial derivatives were done on the field A so they have the momentum as their coefficients

$$H_{EM} = \frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega_p}\Big[(-\omega_p^2+p^2)(a^\dagger_{\vec{-p}}a^\dagger_{\vec{q}} + a_{\vec{p}}a_{\vec{-p}}) + (\omega_p^2+p^2)(a^\dagger_{\vec{p}}a_{\vec{p}} + a_{\vec{p}}a^\dagger_{\vec{p}})\Big] \tag{11.10}$$

The first terms cancel. $\omega_p^2 = p^2$ is again used. And in the last term, the commutation relation $[a_{\vec{p}}, a^\dagger_{\vec{p}}] = (2\pi)^3\delta^{(3)}(\vec{p},\vec{q})$ is used

$$H_{EM} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_p\left(a^\dagger_{\vec{p}}a_{\vec{p}} + a_{\vec{p}}a^\dagger_{\vec{p}}\right) \tag{11.11}$$
$$= \int \frac{d^3p}{(2\pi)^3}\omega_p\left(a^\dagger_{\vec{p}}a_{\vec{p}} + \frac{1}{2}(2\pi)^3\delta^{(3)}(0)\right) \tag{11.12}$$

Where the last term is removed by defining the zero point energy.

$$H_{EM} = \int \frac{d^3p}{(2\pi)^3}\omega_p a^\dagger_{\vec{p}}a_{\vec{p}} \tag{11.13}$$

In [3] they expand the probe's frequency as a constant term plus deviations $\omega_p = \omega_0 + ck$ where ck is shown on figure 11. In quantum field theory this could maybe correspond to transforming a picture where the operators hold the time-dependence related to ω_0. In the rest of this report, we will use the electromagnetic Hamiltonian consistent with [2] and [3]:

$$H_{EM} = \int \frac{d^3p}{(2\pi)^3}\hbar ck\, a^\dagger_{\vec{k}}a_{\vec{k}} \tag{11.14}$$

12 Dipole Interaction in QFT

From arguments of time reversal [4], we know the dipole interaction term in QFT must look like

$$\mathcal{L}_{dip} = -g\bar{\psi}\gamma^{\mu\nu}\gamma^5\psi F_{\mu\nu} \tag{12.1}$$

Where g is the coupling, Ψ is the spinor of the atom and $\gamma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$. Where γ^μ is the 4x4 gamma matrices that are representations of the Clifford algebra and γ^5 is a specific combination of these. In this report I will use the chiral representation:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{12.2}$$

Where every entry is a 2x2 matrix and σ^i are the Pauli matrices.
In (12.1) $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, where A_μ is 4-potential related to electromagnetism. The elements of the field strength tensor is related to the electric and magnetic fields by:

$$E_i = cF_{0i}, \quad B_i = -\frac{1}{2}\epsilon_{ijk}F^{jk} \tag{12.3}$$

This is very convenient for our case, where the atoms do not interact with the magnetic field. This leads all the summations where $\mu = i$ & $\nu = j$ are 0. The remaining summations are computed by finding the $\Psi\gamma^{0i}\gamma^5\Psi$ matrix elements and then letting this act on F_{i0}.
Using that the dual spinor is given by $\bar{\Psi} = \Psi^\dagger\gamma^0$, we get

$$\bar{\psi}(k')\gamma^0\gamma^i\gamma^5\psi(k) = \psi^\dagger(k') \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi(k) \tag{12.4}$$

$$= \psi^\dagger(k') \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} \psi(k) \tag{12.5}$$

Where the chiral representation of the gamma matrices have been used. In this, the spinors can be written as

$$\psi = \frac{1}{2\sqrt{(P'^0 + m)(P^0 + m)}} \begin{pmatrix} P' \cdot \sigma + m \\ P' \cdot \bar{\sigma} + m \end{pmatrix} \mathcal{E}^r \tag{12.6}$$

Where $\bar{\sigma}^\mu = (1, -\sigma^i)$ and $\mathcal{E}^r$ is a two component spinor such that $\mathcal{E}^{r\dagger}\mathcal{E}^r = 1$. Inserting this and using (12.6), we get:

$$bar\Psi(k')\gamma^0\gamma^i\gamma^5\Psi(k) = \frac{1}{2M}\mathcal{E}^{r\dagger}[(P' \cdot \sigma + m,\ P' \cdot \bar{\sigma} + m) \begin{pmatrix} \sigma^i P' \cdot \sigma + m \\ \sigma^i P' \cdot \bar{\sigma} + m) \end{pmatrix}]\mathcal{E}^s \tag{12.7}$$

$$= \frac{1}{2M}\mathcal{E}^{r\dagger}[\tag{12.8}$$

$$-(P'^0 + m)(P^0 + m)\sigma^i + (P'^0 + m)\sigma^i P_j\sigma^j + P'_a\sigma^a\sigma^i(-P^0 - m + P_j\sigma^j) \tag{12.9}$$

$$-(P'^0 + m)(P^0 + m)\sigma^i - (P'^0 + m)\sigma^i P_j\sigma^j - P'_a\sigma^a\sigma^i(-P^0 - m - P_j\sigma^j) \tag{12.10}$$

$$]\mathcal{E}^s \tag{12.11}$$

After removing terms that cancel we get:

$$\frac{1}{2M}\mathcal{E}^{r\dagger}[-2M^2\sigma^i + 2P'_aP_j\sigma^a\sigma^i\sigma^j]\mathcal{E}^s \tag{12.12}$$

$$= M\mathcal{E}^{r\dagger}[-\sigma^i + \frac{P'_iP_J + P'_jP_i}{M^2}\sigma^j + i\epsilon_{aij}\frac{P'_aP_j}{M^2} - \frac{P'_jP_j}{M^2}\sigma^i]\mathcal{E}^s \tag{12.13}$$

All terms with fractions of the type P'_jP_i/M^2 vanish in the non-relativistic limit, since adding the appropriate orders of c yields:

$$\frac{P'_jP_i}{M^2} = \frac{P'_jP_ic^2}{(P'^0c + mc^2)(P^0c + mc^2)} \tag{12.14}$$

$$= \frac{m^2v'_jv_ic^2}{c^4[(m^2 + v'^2/c^2)^{0.5} + m)((m^2 + v^2/c^2)^{0.5} + m]} \approx \frac{v'_jv_i}{4c^2} \tag{12.15}$$

Which is very small in the non-relativistic limit. Cancelling the vanishing terms and using $M \approx 2m$ yields:

$$\bar{\Psi}(k')\gamma^0\gamma^i\gamma^5\Psi(k) \approx -2m\mathcal{E}^{r\dagger}\sigma^i\mathcal{E}^s \tag{12.16}$$

Using $\{\gamma^0, \gamma^i\} = 0$ we know that $\bar{\Psi}\gamma^{0i}\gamma^5\Psi = -\bar{\Psi}\gamma^{i0}\gamma^5\Psi$. With no magnetic interaction, and dividing by 4 from the definition of $\gamma^{\mu\nu}$ this yields:

$$H_{dip} = g\bar{\Psi}\gamma^{\mu\nu}\gamma^5\Psi F_{\mu\nu} = \frac{g}{4}(\bar{\Psi}\gamma^{0i}\gamma^5\Psi F_{0i} + \bar{\Psi}\gamma^{i0}\gamma^5\Psi F_{i0}) \tag{12.17}$$

$$\approx -\frac{1}{2}gm(\mathcal{E}^{r\dagger}\sigma^i\mathcal{E}^sE_i - \mathcal{E}^{r\dagger}\sigma^i\mathcal{E}^s(-E_i)) \tag{12.18}$$

$$= -gm\mathcal{E}^{r\dagger}\sigma^i\mathcal{E}^sE_i \tag{12.19}$$

Since spin and electric dipole moment are both rank one tensors, the Wigner-Eckart theorem [1] says that their expectation values are related by a scalar coefficient. Including this coefficient together with m in the coupling constant s.t. $g = \frac{d_e}{m}$, we have reproduced the usual quantum mechanical dipole operator:

$$H_{dip} = -\mathcal{E}^{r\dagger}\hat{d}_i\,\mathcal{E}^sE_i \tag{12.20}$$

Where $\hat{d}_i$ is the dipole operator along the i'th direction.
No assumption were made as to the form of the spinors Ψ representing the atom, and thus the derivation is a generic derivation of the electric dipole interaction.

13 Deriving Field Theory Hamiltonian for EIT-Setup

The ladder system is commonly used for Rydberg polariton setups [2] [7]. This is shown in figure 11

13.1 Atom Fields

When writing a field theory for particles, one needs to know where the particles are scalar bosons $S = 0$, fermions $S = n/2$ or non-scalar bosons $S = n$. Selection rules of quantum mechanics then assures that a bosonic atom cannot change into a fermionic atom. In the following, the atoms are treated as fermions, since these need the most complete structural description; spinors.
A convenient way of describing the atoms in the ladder system is:

$$\Psi = \begin{pmatrix} \psi_g \\ \psi_- \\ \psi_+ \end{pmatrix} \tag{13.1}$$

Where each of the components of Ψ are of the form (12.7). ψ_+ & ψ_- are related to the old states by a rotation:

$$|+\rangle = \alpha\,|e\rangle + \beta^*\,|r\rangle \tag{13.2}$$
$$|-\rangle = \beta\,|e\rangle - \alpha^*\,|r\rangle \tag{13.3}$$

All these states are Fock states. The ket $|a\rangle$ represents all the atoms at the position of $|a\rangle$. All the atoms but one are always in the ground state. An excited state then corresponds to one of these atoms being excited.
These states are the new eigenstates of the system if only the $|e\rangle - |r\rangle$ coupling Ω is present. These are often called dressed states.

13.2 Dipole Interaction and Effective Transition Fields

Now only the weak coupling g between $|g\rangle - |e\rangle$ needs to be dealt with.Only the combinations of ψ that have non-zero transition dipole moment contribute:

$$H_{dip} = \bar{\Psi}\sigma^{\mu\nu}\gamma^5\Psi F_{\mu\nu} \tag{13.4}$$
$$\approx g[\alpha\bar{\psi_+}\gamma^{i0}\gamma^5\psi_g + \beta\bar{\psi_-}\gamma^{i0}\gamma^5\psi_e + \alpha\bar{\psi_g}\gamma^{i0}\gamma^5\psi_+ + \beta\bar{\psi_e}\gamma^{i0}\gamma^5\psi_-]E_i \tag{13.5}$$

Knowing this, the original Lagrangian can be written compactly using (13.1), using the matrix G that project the relevant fields on each other.

$$\mathcal{L}_{dip} = -gG\bar{\Psi}\gamma^{\mu\nu}\gamma^5\Psi F_{\mu\nu}, \quad G = \begin{pmatrix} 0 & \alpha & \beta \\ \alpha & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix} \tag{13.6}$$

Motivated by this, two new effective fields that describe transitions from $|g\rangle$ to $|+\rangle$ and $|-\rangle$ are defined:

$$b^\dagger = \bar{\psi}_+\gamma^{i0}\gamma^5\psi_g \tag{13.7}$$
$$c^\dagger = \bar{\psi}_-\gamma^{i0}\gamma^5\psi_g \tag{13.8}$$

Showing that b then corresponds to the transition from $|+\rangle \to |g\rangle$

$$b = (b^\dagger)^\dagger = (\bar{\psi}_+ \gamma^{0i} \gamma^5 \psi_g)^\dagger \tag{13.9}$$
$$= \psi_g^\dagger \left(\gamma^0 \gamma^{0i} \gamma^5\right)^\dagger \psi_+ \tag{13.10}$$
$$= \psi_g^\dagger \gamma^5 \gamma^{0i} \gamma^0 \psi_+ \tag{13.11}$$
$$= \psi_g^\dagger \gamma^0 \gamma^{0i} \gamma^5 \psi_+ \tag{13.12}$$
$$= \bar{\psi}_g \gamma^{0i} \gamma^5 \psi_+ \tag{13.13}$$

To get (13.12), both γ^0 and γ^5 were anti-commuted through twice. Now (13.5) can be written as

$$\mathcal{H}_{dip} = g(\alpha b + \alpha b^\dagger + \beta c + \beta c^\dagger) E \tag{13.14}$$

13.3 Free Theory for Atoms as Transition Fields

The free Lagrangian for the atom fields would have a kinetic term and a mass term:

$$\mathcal{L}_A = \Psi_i \gamma^\mu D_\mu \Psi_i + M_{ij} \bar{\Psi}_i \Psi_j \tag{13.15}$$

We assume that the cold atoms are stationary and discard the kinetic term. To write the mass term using the earlier motivated operators $b^\dagger$ and $c^\dagger$, we show that $b^\dagger b \approx \psi_+^\dagger \psi_+$:

$$b^\dagger b = \bar{\psi}_+ \gamma^{0i} \gamma^5 \psi_g \bar{\psi}_g \gamma^{0i} \gamma^5 \psi_+ \tag{13.16}$$
$$= \bar{\psi}_+ \gamma^{0i} \gamma^5 \left(1 - \psi_g^\dagger \psi_g)\right) \gamma^0 \gamma^{0i} \gamma^5 \psi_+ \tag{13.17}$$

Where we have used the fermionic anti-commutation relation $\{\psi_g(x), \psi_g^\dagger(y)\} = \delta(x-y)$ and these b & $b^\dagger$ are on the same point in spacetime. We now use the approximation that almost all the n atoms at position x are in the ground state.

$$\approx \bar{\psi}_+ \gamma^{0i} \gamma^5 \left(1 - n(x)\right) \gamma^0 \gamma^{0i} \gamma^5 \psi_+ \tag{13.18}$$
$$\approx n(x) \bar{\psi}_+ \gamma^{0i} \gamma^5 \gamma^0 \gamma^{0i} \gamma^5 \psi_+ \tag{13.19}$$
$$= -n(x) \bar{\psi}_+ \gamma^0 \gamma^{0i} \gamma^{0i} \gamma^5 \gamma^5 \psi_+ \tag{13.20}$$
$$= -\frac{n(x)}{4} \bar{\psi}_+ \gamma^0 \psi_+ \tag{13.21}$$
$$= -\frac{n(x)}{4} \psi_+^\dagger \psi_+ \tag{13.22}$$

It is now used that the dressed states under the influence of the strong control field Ω have the energies $H_{A+Control} = \Delta_+ \psi_+^\dagger \psi_+ + \Delta_- \psi_-^\dagger \psi_-$ with $\Delta_+ = (\delta + \sqrt{\delta^2 + 4\Omega^2})/2$ & $\Delta_- = (\delta - \sqrt{\delta^2 + 4\Omega^2})/2$ [5]. Using (13.22), this can be written in terms of $b's$ and $c's$. We

redefine the transition operators to include $1/\sqrt{n(x)}$ we get the total Hamiltonian of the EIT ladder setup:

$$\mathcal{H}_0 = \begin{pmatrix} a^\dagger \\ b^\dagger \\ c^\dagger \end{pmatrix}^T \begin{pmatrix} ck & g\alpha\sqrt{n} & g\beta\sqrt{n} \\ g\alpha\sqrt{n} & -\Delta_+ & 0 \\ g\beta\sqrt{n} & 0 & -\Delta_- \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{13.23}$$

Where for the QED creation operators $a,\ a^\dagger$ there is an implicit sum over modes.
Compared to Bienias 2016 [3] this has minuses on the diagonal elements corresponding $b^\dagger b$ & $c^\dagger c$. This is because we assumed the atom fields were fermionic and thus used anticommutation relations in 13.17. For the remainder of this report, I will assume bosonic fields corresponding to pluses on the diagonal.

14 Rydberg Interaction and Interaction Eigenstates

With an eye on modelling the Rydberg polariton interaction, we recreate a Hamiltonian that are more closely related to the eigenstates of the bare atom. These will be the interaction eigenstates of the Rydberg interaction. To describe the Hamiltonian in terms of different states, two unitary matrices($R^{-1}R = 1$) can be inserted on each side of the matrix A in (13.23):

$$\mathcal{H}_0 = \Psi^\dagger_{dressed} R^\dagger R A R^\dagger R \Psi_{dressed} = \Psi'^\dagger B \Psi' \tag{14.1}$$

Where $\Psi' = R\Psi_{dressed}$ are the new states and $B = RAR^\dagger$ is the new quartic Hamilton matrix. Such a transformation can be done between (13.23) and the following equation containing the matrix B:

$$\mathcal{H}_0 = \begin{pmatrix} a^\dagger \\ \psi_s^\dagger \\ \psi_r^\dagger \end{pmatrix}^T \begin{pmatrix} ck & g & 0 \\ g & \delta & \Omega \\ 0 & \Omega & 0 \end{pmatrix} \begin{pmatrix} a \\ \psi_s \\ \psi_r \end{pmatrix} \tag{14.2}$$

Where the transformation matrix R is given by a rotation "around the electric field operators":

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \qquad \tan(2\theta) = \frac{2\Omega}{\delta} \tag{14.3}$$

Since the Ω and g in (14.2) is the couplings between the bare atoms states $|e\rangle$ and $|r\rangle$, the fields must be related to the original states. Thus it is plausible that the Rydberg interaction could be modelled as an interaction between two ψ_r fields. This would mean that the full Hamiltonian including the Rydberg interaction could be written as:

$$H = H_0 + V_{rr}(x-y)\psi_r^\dagger(x)\psi_r^\dagger(y)\psi_r(x)\psi_r(y) \tag{14.4}$$

15 Mass Eigenstates and the Propagator

Mass-eigenstates are the eigenstates of the quadratic part of the Hamiltonian and are the physically observable states. To find these, H_0 is diagonalized using a unitary transformation. As H_0 is symmetric, we know it is diagonalizable. This time starting from (14.2), the transformation is described by the matrices U and $U^\dagger$

$$\mathcal{H}_0 = \Psi^\dagger_{int} U^\dagger U B U^\dagger U \Psi_{int} = \sum_\beta \epsilon_\mu \psi^\dagger_\mu \psi_\mu \tag{15.1}$$

So the mass eigenstates are given by $\psi_\mu = \sum_\alpha U^\alpha_\mu \psi_\alpha$. Where μ runs over $-1, 0, +1$, where $\psi_{\pm 1}$ is interpreted as two bright polaritons and ψ_0 is interpreted as a dark polariton [2]. As the matrix is 3x3, the analytic expression for the eigenvalues and diagonalising matrix are too lengthy to be very onformative.The diagonalising matrix U is quite complex, and same for the eigenvalues ϵ_μ. In Mathematica the eigenvalues and eigenvectors of B can be found and thus also ϵ_μ and P(Since this is just the normalized eigenvectors of B). The expressions are very cumbersome, but the values of ϵ_u as a function of momentum is shown in 12 for $g = 5\delta,\ \Omega = 0.2\delta,\ \delta = 0.04,\ \hbar = c = 1$. We note that these reproduce the results in [2].

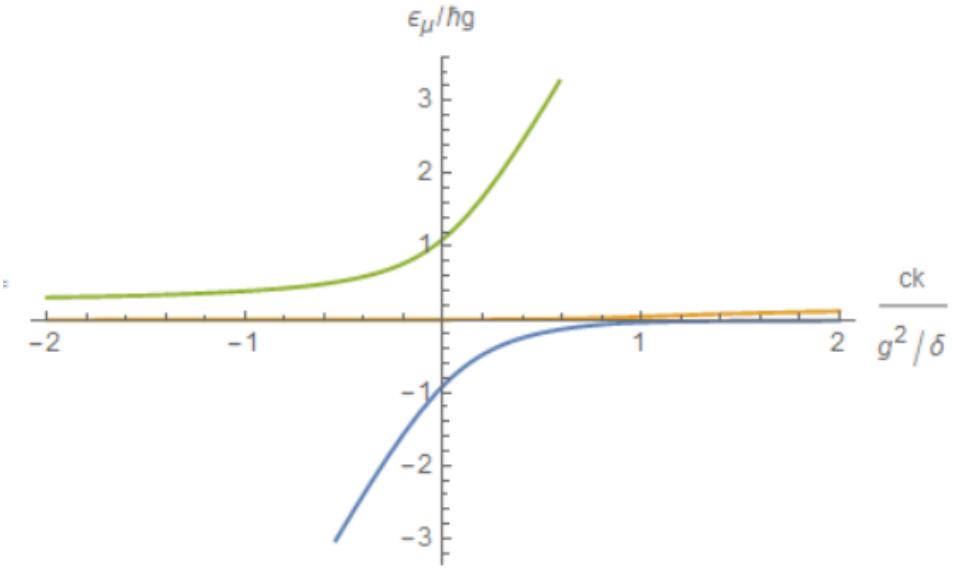


Figure 12: Dispersion relations for ϵ_{+1} (Green), ϵ_0(Orange) and ϵ_{-1} (blue)

As a further test, we expand the eigenvalue of the dark polariton in powers of momentum, to show that we reproduce the expansion in [3].

$$\epsilon_0(k) = \hbar v_g k - \frac{\hbar^2}{2m}k + O(k)^3 \tag{15.2}$$

$$v_g = \frac{\Omega^2}{g^2+\Omega^2}, \quad m = \frac{(g^2+\Omega^2)^3}{2c^2g^2\Omega^2\Delta}\hbar \tag{15.3}$$

Knowing the dispersion relation of the polaritons, the propagator can be computed. In general, the non-relativistic propagator G is defined as:

$$\psi_\mu(\vec{x}', t') = i\int d^3x G(\vec{x}', t', \vec{x}, t)\psi_\mu(\vec{x}, t) \tag{15.4}$$

Assuming G is invariant under space and time transformations, G must be a function of $(x - x')$ & $(t - t')$. A Fourier transform of G is performed:

$$G(x-x', t-t') = \int \frac{d^3k\, d\omega}{(2\pi)^4} e^{i\vec{k}\cdot(\vec{x'-x})} e^{i\omega(t-t')} G(k, \omega) \tag{15.5}$$

Since we are working non-relativistically, we want the propagator to be a Green's function of the free Shrödinger equation. For the mass-eigenstates this means:

$$[i\hbar\partial_t - H_0]\, G(x-x', t-t') = \int \frac{d^3k\, d\omega}{(2\pi)^4} [\hbar\omega - \epsilon_\mu] e^{i\vec{k}\cdot(\vec{x'-x})} e^{i\omega(t-t')} G(k, \omega) \tag{15.6}$$

$$\stackrel{?}{=} \delta(x - x')\delta(t - t') \tag{15.7}$$

The integral over the two exponentials in (15.6) are exactly these delta functions, so $G(k, \omega)$ must be

$$G(k, \omega) = \frac{1}{\hbar\omega - \epsilon_\mu + i\eta} \tag{15.8}$$

Where a small imaginary number $i\eta$ is added to avoid divergences.This is the propagator of a particle under the influence of a Shrödinger equation with no potential. The states that get evolved by this, are the mass eigenstates ie the polaritons. ω was the frequency from the Fourier transform of time, so this will be the total energy ω. Since the interaction eigenstates are the Rydberg states, it would be nice to get the propagator for the Rydberg field operator ψ_r. This will propagate as a linear combination of the three polaritons. What this combination looks like, will be determined by the diagonalizing matrix U. Let us investigate how the Rydberg field with momentum q propagates:

$$\psi_\mu(z', t') = \frac{1}{\hbar\omega - \epsilon_\mu(q) + i\eta} U(q)^s_\mu \psi_s(z, t), \quad \text{No sum over } \mu \tag{15.9}$$

Where the propagator has been given in momentum space. The Fourier transform is left for the Feynman diagrams where it becomes a delta function enforcing momentum conservation

at vertices.
This is now in the polariton basis. To transform back, we apply the inverse transformation matrix $\bar{U}$:

$$\psi_s(z',t') = \sum_\mu \bar{U}_s^\mu(q) \frac{1}{\hbar\omega - \epsilon_\mu(q) + i\eta} U(q)_\mu^s \psi_s(z,t) \tag{15.10}$$

The propagator of ψ_s is now identified

$$\chi_q = \sum_{\mu\in 0,\pm 1} \frac{U_s^\mu(q)U(q)_\mu^s}{\hbar\omega - \epsilon_\mu(q) + i\eta} \tag{15.11}$$

16 Scattering Processes

16.1 Introducing Feynman Diagrams and Scattering amplitudes

In field theory, all processes are computed as scattering processes. A graphical way to keep track of different interactions are the Feynman diagram. In this report, straight lines denote propagators of the mass eigenstates ψ_μ, $\mu \in -1, 0, +1$. Squiggly lines will denote interactions between interaction eigenstates ψ_r.

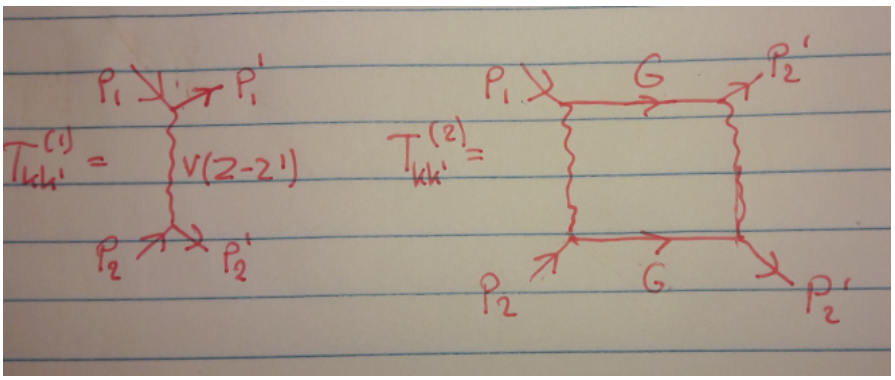


Figure 13: Example of Feynman diagrams. Squiggly lines represent interactions and straight lines represent propagators.

Figure 13 shows two examples of Feynman diagrams. The left is a first order interaction with one order of $V(z-z')$ and no propagators.The 4 tilted "legs" represent the in- and outgoing particles. In momentum space these have initial momentum P_1 & P_2 and final momentum P_1' & P_2'. $V(z-z')$ represents the interaction. When transformed to momentum space $V_{k-k'}$ is dependent on the change in relative momentum between the two particles $k = P_1 - P_2$, $k' = P_1' - P_2'$.
The right is a loop diagram with two orders of $V(z-z')$ and two orders of propagators $G(k,\omega)$. It is called a loop diagram because it forms a loop. The propagators and V is

not fixed by the initial and final momenta. There "runs" a free momentum in the loop, such that all values of momenta are allowed for the different lines, as long as momentum is always conserved at each vertex.
In field theory, the quantity to compute for a scattering process is the quantum amplitude. For a scattering process $|P_1, P_2\rangle \to |P_1', P_2'\rangle$ is given by:

$$\langle P_1', P_2'| U(t, t_0) |P_1, P_2\rangle \tag{16.1}$$

Where $U(t, t_0)$ is the time evolution operator given by Dyson's formula [13]:

$$U(t, t_0) = Texp\left(-i \int_{t_0}^{t} dt' H(t')\right) \tag{16.2}$$

Where T means time ordered and H is the Hamiltonian of the system. When computing this, the exponential is expanded in orders of H. This motivates the categorising of Feynman diagrams into orders of interaction, since this corresponds to the terms in the expansion of $exp(H)$:

$$\langle P_1', P_2'| U(t, t_0) |P_1, P_2\rangle = \langle P_1', P_2'| \left(1 + H + \tfrac{1}{2}H^2 + ...\right) |P_1, P_2\rangle \tag{16.3}$$

The zeroth order term corresponds to nothing happening. It will only contribute if the initial and final states are equal. Therefore it is often omitted from computations. The non-trivial part of the expansion is called the T-matrix. The first term $T^{(1)}$ corresponds to left diagram in 13 and the second term $T^{(2)}$ corresponds to the right diagram. Feynman diagrams represent equations. Written out, the two diagrams in figure 13 read:

$$T^{(1)}_{kk'} = V_{k-k'}, \quad T^{(2)}_{kk'} = T^{(1)}_{kk'} + \int \frac{dq}{2\pi} G(q, \omega)\, V(q + P_1)\, G(q + P_1 + P_2, \omega)\, V(q + P_1) \tag{16.4}$$

Where the labelling of the momentum of inner lines is shown in figure 14. The initial bra and ket have been omitted for shorter notation. The Feynman diagrams really represents $\langle P_1', P_2'| T^{(i)}_{kk'} |P_1, P_2\rangle$.

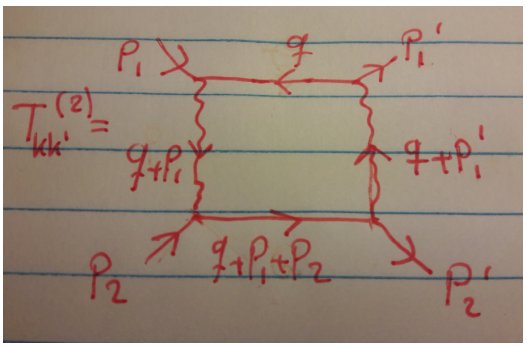


Figure 14: Labelling of inner momentum in loop diagram

16.2 A Single Polariton in an External Potential

A simple case of polariton interaction is the lone polariton experiencing a potential from eg a stationary Rydberg excitation. In this case the interaction Hamiltonian is instead:

$$H_{ext} = \int dz \psi_r(z)^\dagger V(z) \psi_r(z) \tag{16.5}$$

This gives rise to the following types of Feynman diagrams:

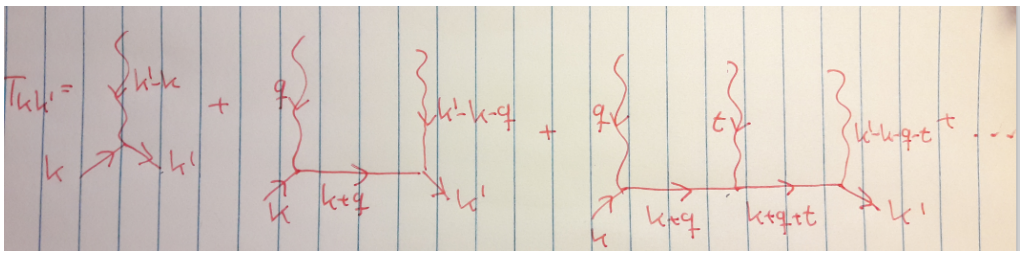


Figure 15: Feynman diagrams for polariton in external potential.(With external legs)

In figure 15 the loose squiggly lines represent the interaction of the polariton with the external potential. In momentum space the first three terms are:

$$T^{(1)}_{kk'} = V(k' - k) \tag{16.6}$$

$$T^{(2)}_{kk'} = V(k' - k) + \int \frac{dq}{2\pi} V(q) \chi_{k+q} V(k' - k - q) \tag{16.7}$$

$$\begin{aligned} T^{(3)}_{kk'} = {} & V(k' - k) + \int \frac{dq}{2\pi} V(q) \chi_{k+q} V(k' - k - q) \\ & + \int \frac{dq}{2\pi} V(q) \chi_{k+q} \int \frac{dt}{2\pi} V(t) \chi_{k+q+t} V(k' - k - q) \end{aligned} \tag{16.8}$$

This comes in a nice recursive form, where each expansion order can be defined in terms of the last:

$$T^{n+1}_{kk'} = V(k' + k) + \int \frac{dq}{2\pi} V(q) \chi_{k+q} T^n_{qk'} \tag{16.9}$$

Since this is an infinite sum, we can take the limit $n \to \infty$. In this limit the equation becomes:

$$T_{kk'} = V(k - k') + \int \frac{dq}{2\pi} V(q) \chi_{k+q} T_{qk'} \tag{16.10}$$

Figure 16 shows this expression diagrammatically. This euqation can be solved for $T_{kk'}$ [3], but this is beyond the scope of this project.

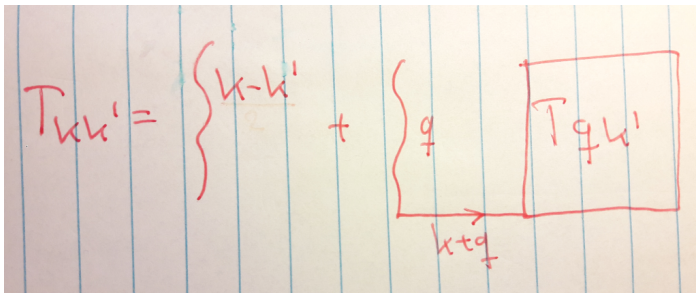


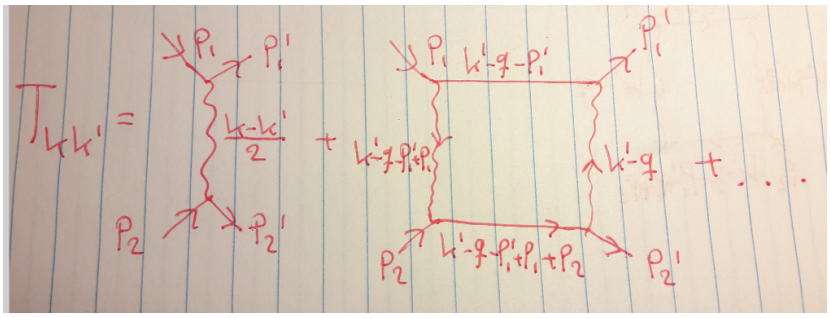
Figure 16: Recursive definition of the T-matrix for external potential. (Without external legs)

16.3 Two Body Interaction

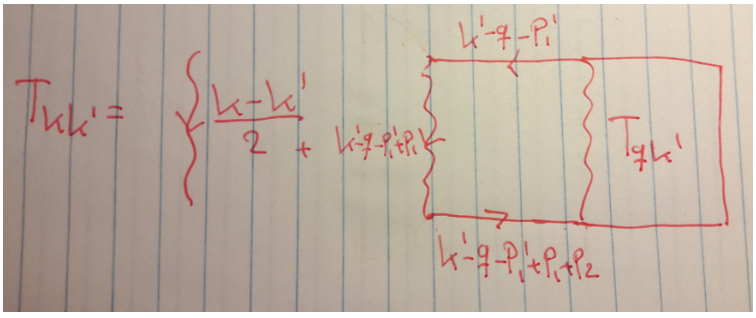
The interaction introduced earlier between two polaritons gives rise to diagrams much like the one described in section 16.2. The interaction term was

$$H_{rr} = V_{rr}(x-y)\psi_r(z_1) \dagger \psi_r(z_2)\psi_r(z_1)\psi_r(z_2) \tag{16.11}$$

The diagrams up to second order is shown in figure 17 together with the diagram describing the recursive formula for the complete T-matrix. k and k' now describe the relative momentum $k = P_1 - P_2$, $k' = P_1' - P_2'$.



(a) First 2 terms



(b) Recursive definition

Figure 17: T-matrix for polariton-polariton interaction

The equation corresponding to this recursive definition will be a Bethe-Salpeter equation:

$$T_{kk'} = V(k-k) + \int \frac{dq}{2\pi} \chi(k'-q-P_1', \omega)\, V(k'-q-P_1'+P_1')\, \chi(k'-q-P_1'+P_1+P_2, \omega)\, T_{qk'} \tag{16.12}$$

Since V is just a number it can be moved to the left of the propagators. Bienias et al 2014 [2] then writes the propagation of the two polaritons as the pair propagator:

$$\chi_q(K, \omega) = \frac{\bar{U}_s^\alpha(p) U_\alpha^s(p) \bar{U}_s^\beta(p') U_\beta^s(p')}{\hbar\omega - \epsilon_\alpha(p) - \epsilon_\beta(p') + i\eta} \tag{16.13}$$

where $p = K/2 + q$ and $p' = K/2 - q$ and K is the total momentum $K = P_1 + P_2 = P_1' + P_2'$. Bienias further rewrites this to the form(see [2] for description of new parameter):

$$\chi_q = \bar{\chi} + \frac{\alpha}{\hbar\bar{\omega} - \hbar^2 q^2/m + i\eta} + \frac{\alpha_B}{\hbar\bar{\omega}_B - \hbar^2 q^2/m + i\eta} \tag{16.14}$$

The first term can be dealt with by including it in the interaction potential s.t.

$$V^{eff}(r) = \frac{V(r)}{1 - \bar{\chi}(\omega) V(r)} \tag{16.15}$$

Where $r = z_1 - z_2$ is the relative position. The second term reduces to a propagator of an combined effective particle. The third term describes resonant scattering, where the incoming polaritons exit as different types. Eg the conversion of two dark polaritons into an upper and lower bright polariton [3]. In the case $\sqrt{|\bar{\omega}\alpha_B^2 \bar{\omega}_B \alpha^2|} \ll 1$ the third term can be ignored and the equation for T becomes:

$$T_{kk'} = V_{k-k'}^{eff} \frac{\alpha}{h\bar{\omega} - \hbar^2 q^2/m} T_{qk'} \tag{16.16}$$

This describes a single particle in an effective potential V^{eff} with mass m(see (16.2)).

$$m = \hbar \frac{(g^2 + \Omega^2)^3}{2c^2 g^2 \Delta\Omega} \tag{16.17}$$

The Shrödinger equation for this particle then captures dynamics of the system:

$$\hbar\bar{\omega}\psi(r) = \left[-\frac{\hbar^2}{m}\partial_r + \alpha V^{eff}(r) \right] \psi(r) \tag{16.18}$$

17 Conclusion

A review of standard EIT in a λ-setup was perfomed and important techniques in AMO-physics were introduced. This includes the modelling of atoms and electric fields, the rotating frame and rotating wave approximation, density operator, susceptibility of coupling fields, the Master equation. Then a symbolic stationary solution of the EIT-setup was found, and a numerical solution of the time evolution was computed.
In the second part of the project, some basic concepts in quantumn field theory was used in the modelling of the electromagnetic fields by QED. A possible term for a dipole-interaction between the atoms and electric field was presented. It was shown that this term reduces to the standard dipole interaction of quantum mechanics in the non-relativistic limit. Then a Hamiltonian of the ladder EIT setup was derived in terms of field operators and transformed to the effective transition field operators. Rydberg polaritons was found to be the mass-eigenstates of the ladder EIT setup. The interaction of Rydberg polaritons with an external field was modelled by a quadratic interaction term. The corresponding Feynman diagrams and the resulting equation for the T-matrix was presented. The interaction of two Rydberg polaritons was modelled by a quartic interaction term and the resulting Feynman diagrams and T-matrix were presented. The resulting system in an appropriate limit were a single particle propagating through an effective potential.

18 Acknowledgements

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References

[1] Gordon Baym and DH Beck, *Elementary quantum mechanics of the neutron with an electric dipole moment*, Proceedings of the National Academy of Sciences **113** (2016), no. 27, 7438–7442.

[2] P Bienias, S Choi, O Firstenberg, MF Maghrebi, M Gullans, Mikhail D Lukin, Alexey Vyacheslavovich Gorshkov, and HP Büchler, *Scattering resonances and bound states for strongly interacting rydberg polaritons*, Physical Review A **90** (2014), no. 5, 053804.

[3] Przemyslaw Bienias, *Few-body quantum physics with strongly interacting rydberg polaritons*, The European Physical Journal Special Topics **225** (2016), no. 15-16, 2957–2976.

[4] JF Donoghue, E Golowich, and BR Holstein, *Dynamics of the standard model*, (1992).

[5] Michael Fleischhauer, Atac Imamoglu, and Jonathan P Marangos, *Electromagnetically induced transparency: Optics in coherent media*, Reviews of modern physics **77** (2005), no. 2, 633.

[6] Alexey V Gorshkov, Johannes Otterbach, Michael Fleischhauer, Thomas Pohl, and Mikhail D Lukin, *Photon-photon interactions via rydberg blockade*, Physical review letters **107** (2011), no. 13, 133602.

[7] MJ Gullans, JD Thompson, Y Wang, Q-Y Liang, V Vuletić, Mikhail D Lukin, and Alexey V Gorshkov, *Effective field theory for rydberg polaritons*, Physical review letters **117** (2016), no. 11, 113601.

[8] Chien Liu, Zachary Dutton, Cyrus H Behroozi, and Lene Vestergaard Hau, *Observation of coherent optical information storage in an atomic medium using halted light pulses*, Nature **409** (2001), no. 6819, 490–493.

[9] David Petrosyan, Johannes Otterbach, and Michael Fleischhauer, *Electromagnetically induced transparency with rydberg atoms*, Phys. Rev. Lett. **107** (2011), 213601.

[10] Jonathan D Pritchard, D Maxwell, Alexandre Gauguet, Kevin J Weatherill, MPA Jones, and Charles S Adams, *Cooperative atom-light interaction in a blockaded rydberg ensemble*, Physical review letters **105** (2010), no. 19, 193603.

[11] M. Saffman, T. G. Walker, and K. Mølmer, *Quantum information with rydberg atoms*, Rev. Mod. Phys. **82** (2010), 2313–2363.

[12] Daniel A. Steck, *Quantum and Atom Optics*, Available online at http://steck.us/teaching (revision 0.8.3, 25 May 2012).

[13] David Tong, *Quantum field theory*, Available online at http://www.damtp.cam.ac.uk/user/tong/qft.html (retrieved 12 June 2017).